The Van-der-Waals Gas EOS for the Lorentz Contracted Rigid Spheres

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Abstract

The relativistic equation of state (EOS) of the Van-der-Waals gas is suggested and analyzed. In contrast to the usual case, the Lorentz contraction of the sphere's volume is taken into account. It is proven that the suggested EOS obeys the causality in the limit of high densities, i.e., the value of sound velocity of such a media is subluminar. The pressure obtained for the high values of chemical potential has an interesting kinetic interpretation. The suggested EOS shows that for high densities the most probable configuration corresponds to the smallest value of the relativistic excluded volume. In other words, for high densities the configurations with the collinear velocities of the neighboring hard core particles are the most probable ones. This, perhaps, may shed light on the coalescence process of any relativistic hard core constituents.

Key words: Equation of state, hard spheres, relativistic Van-der-Waals model

1. Introduction

The van der Waals (VdW) excluded volume model is successfully used to describe the hadron yields measured in relativistic nucleus–nucleus collisions (see e.g. [1, 2] and references therein). This model treats the hadrons as hard core spheres and, therefore, takes into account the hadron repulsion at short distances. In a relativistic situation one should, however, include the Lorentz contraction of the hard core hadrons. Recently, both the conventional cluster and the virial expansions were generalized to the momentum dependent inter-particle potentials, accounting for the Lorentz contracted hard core repulsion [3] and the derived equation of state (EOS) was applied to describe hadron yields observed in relativistic nuclear collisions [4]. The VdW equation obtained in the traditional way leads to the reduction of the second virial coefficient (analog of the excluded volume) compared to nonrelativistic case. However, in the high pressure limit the second virial coefficient remains finite. This fact immediately leads to the problem with causality in relativistic mechanics - the speed of sound exceeds the speed of light [5].

The influence of relativistic effects on the hard core repulsion may be important for a variety of effective models of hadrons and hadronic matter such as the modified Walecka model [6], various extensions of the Nambu–Jona-Lasinio model [7], the quark-meson coupling model [8], the chiral SU(3) model [9] e.t.c. Clearly, the relativistic hard core repulsion should be important for any effective model in which the strongly interacting particles have the reduced values of masses compared to their vacuum values because with lighter masses the large portion of particles becomes relativistic. Nevertheless, the relativistic hard core repulsion was, so far, not incorporated into these models due to the absence of the required formalism.

The Lorentz contraction of rigid spheres representing the hadrons may also be essential at high particle densities which can be achieved at modern colliders. Very recently it was understood that in the baryonless deconfined phase above the cross-over temperature T_c some hadrons may survive up to large temperatures like $3T_c$ [10, 11, 12], and that above T_c there may exist bound states [13] and resonances [14]. Moreover, an exactly solvable statistical model of quark-gluon bags with surface tension [15] indicates that above the cross-over transition [12] the coexistence of hadronic resonances with QGP may, in principle, survive up to infinite temperature. Thus, above T_c the relativistic effects of the hard core repulsion can be important for many hardonic resonances and hadron-like bound states of quarks, especially, if their masses are reduced due to chiral symmetry restoration.

Also the VdW EOS, which obeys the causality condition in the limit of high density and simultaneously reproduces the correct low density behavior, adds a significant theoretical value because such an EOS had not yet been formulated during more than a century of the special relativity. This work is devoted to the investigation of the necessary assumptions to formulate such an equation of state.

The work is organized as follows. In Sect. 2 a summary of both the cluster and virial expansion for the Lorentz contracted rigid spheres is given. It is shown that the VdW extrapolation in relativistic case is not a unique procedure. Therefore, an alternative derivation of the VdW EOS is considered there. The high pressure limit is studied in details in Sect. 3. It is shown that the suggested relativistic generalization of the earlier approach [3] obeys

the causality condition. The conclusions are given in the last section.

2. Relativization of the van der Waals EOS

The excluded volume effect accounts for the blocked volume of two spheres when they touch each other. If hard sphere particles move with relativistic velocities it is necessary to include their Lorentz contraction in the rest frame of the medium. The model suggested in Ref. [16] is not satisfactory: the second virial coefficient $a_2 = 4\, v_{\rm o}$ of the VdW excluded volume model is confused there with the proper volume $v_{\rm o}$ of an individual particle – the contraction effect is introduced for the proper volume of each particle. In order to get the correct result it is necessary to account for the excluded volume of two Lorentz contracted spheres.

Let $\mathbf{r}_i$ and $\mathbf{r}_j$ be the coordinates of the i-th and j-th Boltzmann particle, respectively, and $\mathbf{k}_i$ and $\mathbf{k}_j$ be their momenta, $\hat{\mathbf{r}}_{ij}$ denotes the unit vector $\hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/|\mathbf{r}_{ij}|$ $(\mathbf{r}_{ij} = |\mathbf{r}_i - \mathbf{r}_j|)$. Then for a given set of vectors $(\hat{\mathbf{r}}_{ij}, \mathbf{k}_i, \mathbf{k}_j)$ for the Lorentz contracted rigid spheres of radius $R_{\rm o}$ there exists the minimum distance between their centers $r_{ij}(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j) = \min|\mathbf{r}_{ij}|$. The dependence of the potentials u_{ij} on the coordinates $\mathbf{r}_i, \mathbf{r}_j$ and momenta $\mathbf{k}_i, \mathbf{k}_j$ can be given in terms of the minimal distance as follows

$$u(\mathbf{r}_i, \mathbf{k}_i; \mathbf{r}_j, \mathbf{k}_j) \begin{cases} 0\,, & |\mathbf{r}_i - \mathbf{r}_j| > r_{ij}\,(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j)\ , \\ \\ \infty\,, & |\mathbf{r}_i - \mathbf{r}_j| \le r_{ij}\,(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j)\ . \end{cases} \tag{1}$$

The general approach to the cluster and virial expansions [17] is valid for this momentum dependent potential, and in the grand canonical ensemble it leads to the transcendental equation for pressure [3]

$$p(T,\mu) \;=\; T\rho_t(T)\ \exp\left(\frac{\mu - a_2 p}{T}\right) \;\equiv\; p_{id}(T, \mu - a_2 p)\ , \tag{2}$$

with the second virial coefficient

$$a_2(T) \;=\; \frac{g^2}{\rho_t^2}\int \frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\, e^{-\frac{E(k_1)+E(k_2)}{T}}\; v(\mathbf{k}_1, \mathbf{k}_2)\ , \tag{3}$$

$$v(\mathbf{k}_1, \mathbf{k}_2) \;=\; \frac{1}{2}\int d\mathbf{r}_{12}\ \Theta\left(r_{12}(\hat{\mathbf{r}}_{12}; \mathbf{k}_1, \mathbf{k}_2) \;-\; |\mathbf{r}_{12}|\right)\ , \tag{4}$$

where the thermal density is defined as follows $\rho_t(T) = g\int \frac{d\mathbf{k}}{(2\pi)^3} e^{-\frac{E(k)}{T}}$, degeneracy as g, and $v(\mathbf{k}_1, \mathbf{k}_2)$ denotes the relativistic analog of the usual excluded volume for the two spheres moving with the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and, hence, the factor 1/2 in front of the volume integral in (4) accounts for the fact that the excluded volume of two moving spheres is taken per particle.

In what follows we do not include the antiparticles into consideration to keep it simple, but this can be done easily. Then the pressure (2) generates the following particle density

$$n(T,\mu) = \frac{\partial p(T,\mu)}{\partial \mu} = \frac{e^{\frac{\mu}{T}}\rho_t(T)}{1 + e^{\frac{\mu}{T}}\rho_t(T) a_2(T)} \equiv \frac{p}{T\left(1 + e^{\frac{\mu}{T}}\rho_t(T) a_2(T)\right)}\,, \tag{5}$$

which in the limit of high pressure $p(T,\mu) \rightarrow \infty$ gives a limiting value of particle density $n(T,\mu) \rightarrow a_2^{-1}(T)$.

A form of Eq. (2) with constant a_2 was obtained for the first time in Ref. [6]. The new feature of Eq. (2) is the temperature dependence of the excluded volume a_2 (T) (3) which is due to the Lorentz contraction of the rigid spheres. This is a necessary and important modification which accounts for the relativistic properties of the interaction. It leads, for instance, to a 50 % reduction of the excluded volume of pions already at temperatures $T = 140$ MeV [3].

The calculation of the cluster integral in relativistic case is more complicated because each sphere becomes an ellipsoid due to the Lorentz contraction and because the relativistic excluded volume strongly depends not only on the contraction of the spheres, but also on the angle between the particle 3-velocities. Therefore, in Appendix A we give a derivation of a rather simple formula for the coordinate space integration in a_2 which is found to be valid with an accuracy of a few percents for all temperatures. Its simplicity enables us to perform the angular integrations in $a_2(T)$ analytically and obtain

$$a_2(T) \approx \frac{\alpha v_o}{8}\left(3\pi + \frac{74\,\rho_s}{3\,\rho_t}\right) \;, \quad \rho_s(T) = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{m}{E}\;e^{-\frac{E}{T}} \;. \tag{6}$$

The expression for the coefficient $\alpha \approx 1/1.065$ is given in Appendix A by Eq. (51). Using this result it is easy to show that in the limit of high temperature $T \gg m$ the ratio of the scalar density $\rho_s(T)$ to the thermal density $\rho_t(T)$ in (6) vanishes and the second virial coefficient approaches the constant value:

$$a_2(T)\Big|_{T\gg m} \longrightarrow \;\frac{3\pi\alpha v_o}{8} + \mathrm{O}\left(\frac{m}{T}\right) \;, \tag{7}$$

which is about $\frac{3\pi}{32}$ times smaller compared to the value of the nonrelativistic excluded volume, and, hence, is surprisingly very close to the dense packing limit of the nonrelativistic hard spheres. Similarly to the nonrelativistic VdW case [5] this leads to the problem with causality at very high pressures. Of course, in this formulation the superluminar speed of sound should appear at very high temperatures which are unreachable in hadronic phase. Thus the simple "relativization" of the virial expansion is much more realistic than the nonrelativistic description used in Refs. [1, 2], but it does not solve the problem completely.

The reason why the simplest generalization (2) fails is rather trivial. Eq. (2) does not take into account the fact that at high densities the particles disturb the motion of their neighbors. The latter leads to the more compact configurations than predicted by Eqs. (2 - 4), i.e., the motion of neighboring particles becomes correlated due to a simple geometrical reason. In other words, since the N-particle distribution is a monotonically decreasing function of the excluded volume, the most probable state should correspond to the configurations of smallest excluded volume of all neighboring particles. This subject is, of course, far beyond the present paper. Although we will touch this subject slightly while discussing the limit $\mu/T \gg 1$ in Sect. 3, our primary task here will be to give a relativistic generalization of the VdW EOS, which at low pressures behaves in accordance with the relativistic virial expansion presented above, and at the same time is free of the causality paradox at high pressures.

In our treatment, we will completely neglect the angular rotations of the Lorentz contracted spheres because their correct analysis can be done only within the framework of quantum scattering theory which is beyond the current scope. However, it is clear that the rotational effects can be safely neglected at low densities because there are not so many collisions in the system. At the same time the rotations of the Lorentz contracted spheres at very high pressures, which are of the principal interest, can be neglected too, because at so high densities the particles should be so close to each other, that they must prevent the rotations of neighboring particles. Thus, for these two limits we can safely ignore the rotational effects and proceed further on like for the usual VdW EOS.

Eq. (2) is only one of many possible VdW extrapolations to high density. As in non-relativistic case, one can write many expressions which will give the first two terms of the full virial expansion exactly, and the difference will appear in the third virial coefficient. In relativistic case there is an additional ambiguity: it is possible to perform the momentum integration, first, and make the VdW extrapolation next, or vice versa. The result will, evidently, depend on the order of operation.

As an example let us give a brief "derivation" of Eq. (2), and its counterpart in the grand canonical ensemble. The two first terms of the standard cluster expansion read as [17, 3]

$$p = T\,\rho_t(T)\;e^{\frac{\mu}{T}}\left(1 - a_2\,\rho_t(T)\,e^{\frac{\mu}{T}}\right)\;. \tag{8}$$

Now we approximate the last term on the right hand side as $\rho_t(T)\,e^{\frac{\mu}{T}} \approx \frac{p}{T}$. Then we extrapolate it to high pressures by moving this term into the exponential function as

$$p \approx T\,\rho_t(T)\;e^{\frac{\mu}{T}}\left(1 - a_2\,\frac{p}{T}\right) \approx T\,\rho_t(T)\;\exp\left(\frac{\mu - a_2\,p}{T}\right)\;. \tag{9}$$

The resulting expression coincides with Eq. (2), but the above manipulations make it simple and transparent. Now we will repeat all the above steps while keeping both momentum integrations fixed

$$\begin{aligned} p \;&\approx\; \frac{T\,g^2\,e^{\frac{\mu}{T}}}{\rho_t(T)}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\;e^{-\frac{E(k_1)+E(k_2)}{T}}\left(1 - \frac{v(\mathbf{k}_1,\mathbf{k}_2)\,p}{T}\right) \\ &\approx\; \frac{T\,g^2}{\rho_t(T)}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\;e^{\frac{\mu - v(\mathbf{k}_1,\mathbf{k}_2)\,p\,-\,E(k_1)\,-\,E(k_2)}{T}}\;. \end{aligned} \tag{10}$$

The last expression contains the relativistic excluded volume (4) explicitly and, as can be shown, is free of the causality paradox. This is so because at high pressures the main contribution to the momentum integrals corresponds to the smallest values of the excluded volume (4). It is clear that such values are reached when the both spheres are ultrarelativistic and their velocities are collinear.

With the help of the following notations for the averages

$$\langle\mathcal{O}\rangle \;\equiv\; \frac{g}{\rho_t(T)}\int\frac{d\mathbf{k}}{(2\pi)^3}\;\mathcal{O}\;e^{-\frac{E(k)}{T}}\,, \tag{11}$$

$$\langle\langle\mathcal{O}\rangle\rangle \;\equiv\; \frac{g^2}{\rho_t^2(T)}\int\frac{d\mathbf{k}_1}{(2\pi)^3}\frac{d\mathbf{k}_2}{(2\pi)^3}\;\mathcal{O}\;e^{-\frac{v(\mathbf{k}_1,\mathbf{k}_2)\,p\,+\,E(k_1)\,+\,E(k_2)}{T}}\,, \tag{12}$$

we can define all other thermodynamic functions as

$$
\begin{aligned}
n(T,\mu) &= \frac{\partial p(T,\mu)}{\partial \mu} = \frac{p}{T\left(1 + e^{\frac{\mu}{T}}\rho_t(T)\langle\langle v(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle\right)}\,, && (13)\\
s(T,\mu) &= \frac{\partial p(T,\mu)}{\partial T} = \frac{p}{T} + \frac{1}{T}\frac{\left(2\,e^{\frac{\mu}{T}}\rho_t(T)\langle\langle E\rangle\rangle - [\mu + \langle E\rangle]\,p\,T^{-1}\right)}{1 + e^{\frac{\mu}{T}}\rho_t(T)\langle\langle v(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle}\,, && (14)\\
\varepsilon(T,\mu) &= T\,s(T,\mu) + \mu\, n(T,\mu) - p(T,\mu) = \frac{2\,e^{\frac{\mu}{T}}\rho_t(T)\langle\langle E\rangle\rangle - [\mu + \langle E\rangle]\,p\,T^{-1}}{1 + e^{\frac{\mu}{T}}\rho_t(T)\langle\langle v(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle}\,. && (15)
\end{aligned}
$$

Here $n(T,\mu)$ is the particle density, while $s(T,\mu)$ and $\varepsilon(T,\mu)$ denote the entropy and energy density, respectively.

In the low pressure limit $4\,p\,v_o T^{-1} \ll 1$ the corresponding exponent in (12) can be expanded and the mean value of the relativistic excluded volume can be related to the second virial coefficient $a_2(T)$ as follows

$$
\langle\langle v(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle \approx a_2(T) \;-\; \frac{p}{T}\langle\langle v^2(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle\,, \qquad (16)
$$

which shows that at low pressures the average value of the relativistic excluded volume should match the second virial coefficient $a_2(T)$, but should be smaller than $a_2(T)$ at higher pressures and this behavior is clearly seen in Fig. 1.

A comparison of the particle densities (5) and (13) shows that despite the different formulae for pressure the particle densities of these models have a very similar expression, but in (13) the second virial coefficient is replaced by the averaged value of the relativistic excluded volume $\langle\langle v(\mathbf{k}_1,\mathbf{k}_2)\rangle\rangle$. Such a complicated dependence of the particle density (13) on T and μ requires a nontrivial analysis for the limit of high pressures.

To analyze the high pressure limit $p \to \infty$ analytically we need an analytic expression for the excluded volume. For this purpose we will use the ultrarelativistic expression derived in the Appendix A:

$$
v(\mathbf{k}_1,\mathbf{k}_2) \approx \frac{v_{12}^{Urel}(R,R)}{2} \equiv \frac{v_o}{2}\left(\frac{m}{E(\mathbf{k}_1)} + \frac{m}{E(\mathbf{k}_2)}\right)\left(1 + \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 + \frac{3\,v_o}{2}\sin\left(\Theta_v\right)\;. \qquad (17)
$$

As usual, the total excluded volume $v_{12}^{Urel}(R,R)$ is taken per particle. Eq. (17) is valid for $0 \le \Theta_v \le \frac{\pi}{2}$; to use it for $\frac{\pi}{2} \le \Theta_v \le \pi$ one has to make a replacement $\Theta_v \longrightarrow \pi - \Theta_v$ in (17). Here the coordinate system is chosen in such a way that the angle Θ_v between the 3-vectors of particles' momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ coincides with the usual spherical angle Θ of spherical coordinates (see Appendix A). To be specific, the OZ-axis of the momentum space coordinates of the second particle is chosen to coincide with the 3-vector of the momentum $\mathbf{k}_1$ of the first particle.

The Lorentz frame is chosen to be the rest frame of the whole system because otherwise the expression for pressure becomes cumbersome. Here v_o stands for the eigen volume of particles which, for simplicity, are assumed to have the same hard core radius and the same mass.

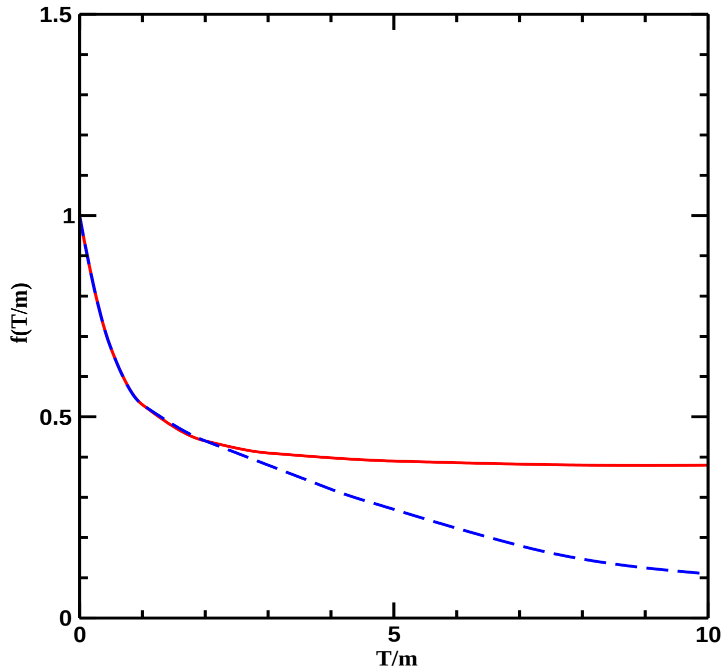

Fig. 1. Comparison of the exact value of the second virial coefficient $a_2(T)/a_2(0)$ (solid curve) with the averaged value of the relativistic excluded volume $\alpha\langle\langle v(\mathbf{k}_1, \mathbf{k}_2)\rangle\rangle/(a_2(0))$ (dashed curve) given by Eq. (17) fot $\mu = 0$. The normalization coefficient $\alpha \approx 1/1.065$ (51) is introduced to reproduce the low density results.

Despite the fact that this equation was obtained for ultrarelativistic limit, it is to a within few per cent accurate in the whole range of parameters (see Fig. 1 and Appendix A for the details), and, in addition, it is sufficiently simple to allow the analytical treatment.

3. High Pressure Limit

As seen from the expression for the relativistic excluded volume (17) , for very high pressures only the smallest values of the relativistic excluded volume will give a non-vanishing contribution to the angular integrals of thermodynamic functions. This means that only Θ_v-values around 0 and around π will contribute into the thermodynamic functions (see Fig. 2). Using the variable $x = \sin^2(\Theta_v/2)$, one can rewrite the $\mathbf{k}_2$ angular integration as follows

$$I_\Theta(k_1) = \int \frac{d\mathbf{k}_2}{(2\pi)^3} e^{-\frac{v(\mathbf{k}_1,\mathbf{k}_2)p}{T}} 4 \int \frac{d\,k_2 k_2^2}{(2\pi)^2} \int_0^{0.5} d\,x\; e^{-\left(AC\left(1-\frac{x}{2}\right)^2 + B\sqrt{x(1-x)}\right)} \,, \tag{18}$$

$$\text{with} \quad A = 2v_o \frac{p}{T}\,; \quad B = \frac{3}{2}A\,; \quad C = \left(\frac{m}{E(k_1)} + \frac{m}{E(k_2)}\right) \,, \tag{19}$$

where we have accounted for the fact that the integration over the polar angle gives a factor 2π and that one should double the integral value in order to integrate over a half of the Θ_v range.

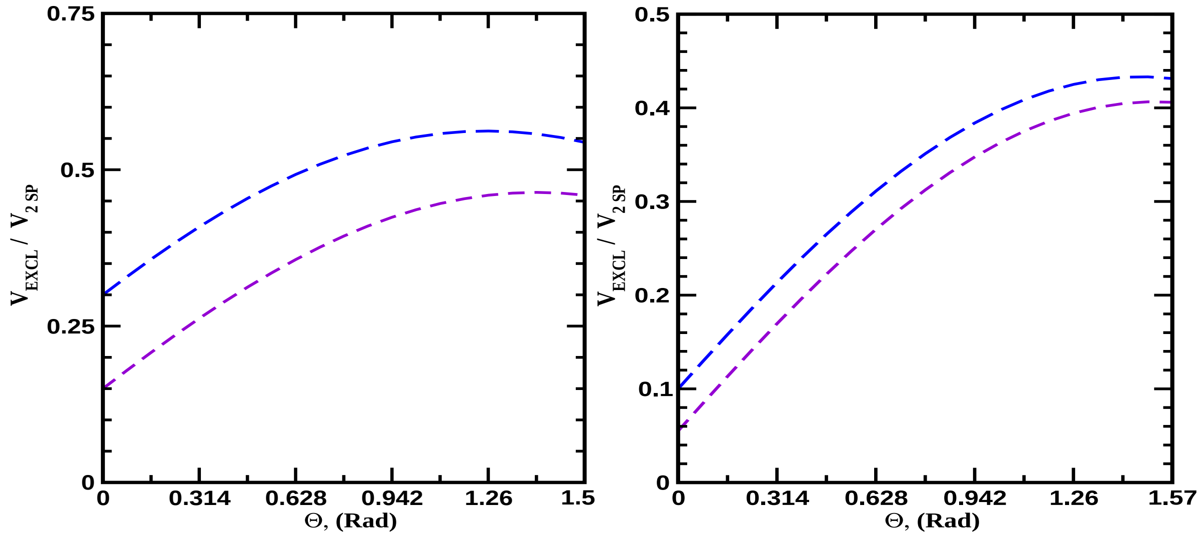

Fig. 2. Comparison of the relativistic excluded volumes for highly contracted spheres. In the left panel the long dashed curve corresponds to $\frac{E(k_1)}{m} = 2$ and $\frac{E(k_2)}{m} = 10$ whereas the short dashed curve is found for $\frac{E(k_1)}{m} = 5$ and $\frac{E(k_2)}{m} = 10$. The corresponding values in the right panel are $\frac{E(k_1)}{m} = 10$, $\frac{E(k_2)}{m} = 10$ (long dashed curve) and $\frac{E(k_1)}{m} = 10$, $\frac{E(k_2)}{m} = 100$ (short dashed curve). It shows that the excluded volume for Θ_v close to $\frac{\pi}{2}$ is finite always, while for the collinear velocities the excluded volume approaches zero, if both spheres are ultrarelativistic.

Since $C \leq 2$ in (19) is a decreasing function of the momenta, then in the limit $A \gg 1$ one can account only for the $\sqrt{x}$ dependence in the exponential in (18) because it is the leading one. Then integrating by parts one obtains

$$I_\Theta(k_1) \approx 4 \int \frac{d\,k_2 k_2^2}{(2\pi)^2}\; e^{-AC} \int_0^{0.5} d\,x\; e^{-B\sqrt{x}} \approx 8 \int \frac{d\,k_2 k_2^2}{(2\pi)^2}\; e^{-AC} \frac{1}{B^2}\;. \tag{20}$$

Applying the above result to the pressure (10), in the limit under consideration one finds that the momentum integrals are decoupled and one gets the following equation for pressure

$$p(T,\mu) \approx \frac{16\, T^3 e^{\frac{\mu}{T}}}{9\, v_o^2\, p^2\, \rho_t(T)} \left[g \int \frac{d\,k k^2}{(2\pi)^2}\; e^{-\frac{E(k)}{T} \;-\; \frac{2\,v_o\, m}{T E(k)}\, p} \right]^2 . \tag{21}$$

Now it is clearly seen that at high pressures the momentum distribution function in (21) may essentially differ from the Boltzmann one. To demonstrate this we can calculate an effective temperature by differentiating the exponential under the integral in (21) with respect to particle's energy E:

$$T_{eff}(E) \;=\; - \left[\frac{\partial}{\partial E} \left(- \frac{E}{T} - \frac{2\, v_o\, m}{T E}\, p \right) \right]^{-1} \;=\; \frac{T}{1 \;-\; \frac{2\, v_o\, m}{E^2} p}\;. \tag{22}$$

Eq. (22) shows that the effective temperature $T_{eff}(E \to \infty) = T$ may be essentially lower than that one at $E = m$. In fact, at very high pressures the effective temperature $T_{eff}(m)$ may become negative.

A sizable difference between T_{eff} values at high and low particle energies mimics the collective motion of particles since a similar behavior is typical for the transverse energy spectra of particles having the collective transverse velocity which monotonically grows with the transverse radius [18, 19]. However, in contrast to the true collective motion case [18, 19] , the low energy T_{eff} (22) gets higher for smaller masses of particles. Perhaps, such a different behavior of low energy effective temperatures can be helpful for distinguishing the high pressure case from the collective motion of particles.

Our next step is to perform the gaussian integration in Eq. (21). Analyzing the function

$$F \equiv 2 \ln k - \frac{E(k)}{T} - A \frac{m}{E(k)} \tag{23}$$

for $A \gg 1$, one can safely use the ultrarelativistic approximation for particle momenta $k \approx E(k) \rightarrow \infty$. Then it is easy to see that the function F in (23) has an extremum at

$$\frac{\partial F}{\partial E} = \frac{2}{E} - \frac{1}{T} + A \frac{m}{E^2} = 0 \quad \Rightarrow \quad E = E^* \approx \frac{A\, m}{\sqrt{1 + \frac{A\, m}{T}} - 1} \equiv T \left(\sqrt{1 + \frac{A\, m}{T}} + 1 \right) , \tag{24}$$

which turns out to be a maximum, since the second derivative of F (23) is negative

$$\left. \frac{\partial^2 F}{\partial E^2} \right|_{E=E^*} \approx - \frac{2}{(E^*)^2} - 2\, A \frac{m}{(E^*)^3} < 0\, . \tag{25}$$

There are two independent ways to increase pressure: one can increase the value of chemical potential while keeping temperature fixed and vice versa. We will consider the high chemical potential limit $\mu/T \gg 1$ for finite T first, since this case is rather unusual. In this limit the above expressions can be simplified further on

$$E^* \approx \sqrt{2\, m\, v_o\, p}\, , \quad \Rightarrow \quad \left. \frac{\partial^2 F}{\partial E^2} \right|_{E=E^*} \approx - \frac{2}{T \sqrt{2\, m\, v_o\, p}}\, . \tag{26}$$

Here in the last step we explicitly substituted the expression for A. Performing the gaussian integration for momenta in (21), one arrives at

$$\int \frac{d\, k k^2}{(2\pi)^2}\, e^{- \frac{E(k)}{T} - \frac{2\, v_o\, m}{T E(k)}\, p} \approx \frac{(E^*)^2}{(2\pi)^2} \sqrt{\pi\, T\, E^*}\, e^{- \frac{2\, E^*}{T}}\, , \tag{27}$$

which leads to the following equation for the most probable energy of particle E^*

$$E^* \approx D\, T^4\, e^{\frac{\mu - 4 E^*}{T}}\, , \quad D \equiv \frac{8\, g^2 m^3 v_o}{9\, \pi^3 \rho_t(T)}\, . \tag{28}$$

As one can see, Eq. (28) defines pressure of the system. Close inspection shows that the high pressure limit can be achieved, if the exponential in (28) diverges much slower than μ/T. The latter defines the EOS in the leading order as

$$E^* \approx \frac{\mu}{4}\, , \quad \Rightarrow \quad p \approx \frac{\mu^2}{32\, m\, v_o}\, . \tag{29}$$

The left hand side equation above demonstrates that in the $\mu/T \gg 1$ limit the natural energy scale is given by a chemical potential. This is a new and important feature of the relativistic VdW EOS compared to the previous findings.

The right hand side Eq. (29) allows one to find all other thermodynamic functions in this limit from thermodynamic identities:

$$s \approx 0\,, \quad n \approx \frac{2\,p}{\mu}\,, \quad \varepsilon \equiv Ts + \mu n - p \approx p\,. \tag{30}$$

Thus, we showed that for $\mu/T \gg 1$ and finite T the speed of sound c_s in the leading order does not exceed the speed of light since

$$c_s^2 = \left.\frac{\partial p}{\partial \varepsilon}\right|_{s/n} = \frac{d\,p}{d\,\varepsilon} = 1\,. \tag{31}$$

From Eq. (28) it can be shown that the last result holds in all orders.

It is interesting that the left hand side Eq. (26) has a simple kinetic interpretation. Indeed, recalling that the pressure is the change of momentum during the collission time one can write (24) as follows (with $E^* = k^*$)

$$p = \frac{(k^*)^2}{2\,m\,v_{\rm o}} = \frac{2\,k^*}{\pi R_{\rm o}^2} \cdot \frac{3\,v^*\gamma^*}{8\,R_{\rm o}} \cdot \frac{1}{2}\,. \tag{32}$$

In the last result the change of momentum during the collision with the wall is $2\,k^*$, which takes the time $\frac{8\,R_{\rm o}}{3\,v^*\gamma^*}$. The latter is twice of the Lorentz contracted height $(4/3R_{\rm o})$ of the cylinder of the base $\pi R_{\rm o}^2$ which is passed with the speed v^*. Here the particle velocity v^* and the corresponding gamma-factor γ^* are defined as $v^*\gamma^* = k^*/m$. The rightmost factor $1/2$ in (32) accounts for the fact that only a half of particles moving perpendicular to the wall has the momentum $-k^*$. Thus, Eq. (32) shows that in the limit under consideration the pressure is generated by the particle momenta which are perpendicular to the wall. This, of course, does not mean that all particles in the system have the momenta which are perpendicular to a single wall. No, this means that in those places near the wall where the particles' momenta are not perpendicular (but are parallel) to it, the change of momentum $2k^*$ is transferred to the wall by the particles located in the inner regions of the system whose momenta are perpendicular to the wall. Also it is easy to deduce that such a situation is possible, if the system is divided into the rectangular cells or boxes inside which the particles are moving along the height of the box and their momenta are collinear, but they are perpendicular to the particles' momenta in all surrounding cells. Note that appearing of particles' cells is a typical feature of the treatment of high density limit [20] and can be related to a complicated phase structure of nuclear matter at very low temperatures [21].

Of course, inside of such a box each Lorentz contracted sphere would generate an excluded volume which is equal to a volume of a cylinder of height $\frac{2\,R_{\rm o}}{\gamma^*}$ and base $\pi R_{\rm o}^2$. This cylinder, of course, differs from the cylinder involved in Eq. (32), but we note that exactly the hight $\frac{4R_{\rm o}}{3\,\gamma^*}$ is used in the derivation of the ultrarelativistic limit for the relativistic excluded volume (50) (see Appendix A for details). Thus, it is very interesting that in contrast to nonrelativistic case the relativistic excluded volume $\frac{4\pi R_{\rm o}^3}{3\,\gamma^*}$ which enters into Eq. (32) is only

33 % smaller than the excluded volume $\frac{2\pi R_o^3}{\gamma^*}$ of ultrarelativistic particle at high pressures. Also it is remarkable that the low density EOS extrapolated to very high values of the chemical potential, at which it is not supposed to be valid at all, gives a reasonable estimate for the pressure at high densities.

Another interesting conclusion that follows from this limit is that for the relativistic VdW systems existing in the nonrectangular volumes the relativistic analog of the dense packing may be unstable.

The analysis of the limit $T/\mu \gg 1$ and finite μ also starts from Eqs. (21)–(24). The function F from (23) again has the maximum at $E^* \equiv E(k^*) = k^*$ defined by the right hand side Eq. (24). Now the second derivative of function F becomes

$$\left.\frac{\partial^2 F}{\partial E^2}\right|_{E=E^*} \approx -\frac{2}{(E^*)^2} - 2\,A\frac{m}{(E^*)^3} = -\frac{2\sqrt{1+\frac{A\,m}{T}}}{(E^*)^2}\,. \tag{33}$$

This result allows one to perform the gaussian integration for momenta in (21) for this limit and get

$$\int \frac{d\,k k^2}{(2\pi)^2}\, e^{-\frac{E(k)}{T} - \frac{2\,v_o\,m}{TE(k)}\,p} \approx \frac{(E^*)^3\, e^{-2\left(1+\frac{A\,m}{T}\right)^{\frac{1}{2}}}}{(2\pi)^2\left(1+\frac{A\,m}{T}\right)^{\frac{1}{4}}}\, I_\xi\left(1+\frac{A\,m}{T}\right)\,, \tag{34}$$

where the auxiliary integral I_ξ is defined as follows

$$I_\xi(x) \equiv \int\limits_{-x^{\frac{1}{4}}}^{+\infty} d\xi\; e^{-\xi^2}\,. \tag{35}$$

The expression (34) can be also used to find the thermal density $\rho_t(T)$ in the limit $T \to \infty$ by the substitution $A = 0$. Using (34), one can rewrite the equation for pressure (21) as the equation for the unknown variable $z \equiv A\,m/T \equiv \frac{2\,v_o\,m\,p}{T^2}$

$$z^3 \approx e^{\frac{\mu}{T}}\phi(z)\,, \quad \phi(z) \equiv \frac{2\,g\,v_o\,m^3\,I_\xi^2(1+z)\left(1+(1+z)^{\frac{1}{2}}\right)^6}{\left(3\,\pi\, e^{2\cdot\sqrt{1+z}-1}\right)^2 I_\xi(1)(1+z)^{\frac{1}{2}}}\,. \tag{36}$$

Before continuing our analysis further on, it is necessary to make two comments concerning Eq. (36). First, rewriting the left hand side Eq. (36) in terms of pressure, one can see that the value of chemical potential is formally reduced exactly in three times. In other words, it looks like that in the limit of high temperature and finite μ the pressure of the relativistic VdW gas is created by the particles with the charge being equal to the one third of their original charge. Second, due to the nonmonotonic dependence of $\phi(z)$ in the right hand side Eq. (36) it is possible that the left hand side Eq. (36) can have several solutions for some values of parameters. Leaving aside the discussion of this possibility, we will further consider only such a solution of (36) which corresponds to the largest value of the pressure (21).

Since the function $\phi(z)$ does not have any explicit dependence on T or μ, one can establish a very convenient relation

$$\frac{\partial z}{\partial T} = -\frac{\mu}{T}\frac{\partial z}{\partial \mu} \tag{37}$$

between the partial derivatives of z given by the left hand side Eq. (36). Using (37), one can calculate all the thermodynamic functions from the pressure $p = \beta T^2 z$ (with $\beta \equiv (3\, m\, v_o)^{-1}$) as follows:

$$\begin{aligned} n &\approx \beta T^2 \frac{\partial z}{\partial \mu}\,, && (38)\\ s &\approx \beta \left[2\, T\, z + T^2 \frac{\partial z}{\partial T} \right] = \frac{2\, p - \mu n}{T}\,, && (39)\\ \varepsilon &\equiv T s + \mu n - p \approx p\,. && (40) \end{aligned}$$

The last result leads to the causality condition (31) for the limit $T/\mu \gg 1$ and finite μ.

In fact, the above result can be extended to any $\mu > -\infty$ and any value of T satisfying the inequality

$$E^* \approx T \left(\sqrt{1+z} + 1 \right) \gg m\,, \qquad (41)$$

which is sufficient to derive Eq. (36). To show this, it is sufficient to see that for $z = 0$ there holds the inequality $z^3 < e^{\frac{\mu}{T}} \phi(z)$, which changes to the opposite inequality $z^3 > e^{\frac{\mu}{T}} \phi(z)$ for $z = \infty$. Consequently, for any value of μ and T satisfying (41) the left hand side Eq. (36) has at least one solution $z^* > 0$ for which one can establish Eqs. (37)–(40) and prove the validity of the causality condition (31).

The model (10) along with the analysis of high pressure limit can be straightforwardly generalized to include several particle species. For the pressure $p(T, \{\mu_i\})$ of the mixture of N-species with masses m_i ($i = \{1, 2, .., N\}$), degeneracy g_i, hard core radius R_i and chemical potentials μ_i is defined as a solution of the following equation

$$p(T, \{\mu_i\}) = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \sum_{i,j=1}^{N} \frac{T\, g_i\, g_j}{\rho_{tot}(T, \{\mu_l\})}\, e^{\frac{\mu_i + \mu_j - v_{ij}(\mathbf{k}_1, \mathbf{k}_2)\, p - E_i(k_1) - E_j(k_2)}{T}}\,, \qquad (42)$$

where the relativistic excluded volume per particle of species i (with the momentum $\mathbf{k}_1$) and j (with the momentum $\mathbf{k}_2$) is denoted as $v_{ij}(\mathbf{k}_1, \mathbf{k}_2)$, $E_i(k_1) \equiv \sqrt{k_1^2 + m_i^2}$ and $E_j(k_2) \equiv \sqrt{k_2^2 + m_j^2}$ are the corresponding energies, and the total thermal density is given by the expression

$$\rho_{tot}(T, \{\mu_i\}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{i=1}^{N} g_i\, e^{\frac{\mu_i - E_i(k)}{T}}\,. \qquad (43)$$

The excluded volume $v_{ij}(\mathbf{k}_1, \mathbf{k}_2)$ can be accurately approximated by $\alpha\, v_{12}^{Urel}(R_i, R_j)/2$ defined by Eqs. (50) and (51).

The multicomponent generalization (42) is obtained in the same sequence of steps as the one-component expression (10). The only difference is in the definition of the total thermal density (43) which now includes the chemical potentials. Note also that the expression (42) by construction recovers the virial expansion up to the second order at low particle densities, but it cannot be reduced to any of two extrapolations which are suggested in [22] and [23] for the multicomponent mixtures and carefully analyzed in Ref. [4]. Thus, the expression (42) removes the non-uniqueness of the VdW extrapolations to high densities, if one requires a causal behavior in this limit.

4. Concluding Remarks

In this work we proposed a relativistic analog of the VdW EOS which reproduces the virial expansion for the gas of the Lorentz contracted rigid spheres at low particle densities and is causal at high densities. As one can see from the expression for particle density (13) and from the corresponding relation for effective temperature (22) the one-particle momentum distribution function has a more complicated energy dependence than the usual Boltzmann distribution function, which would be interesting to check experimentally. Such a task involves considerable technical difficulties since the particle spectra measured in high energy nuclear collisions involve a strong collective flow which can easily hide or smear the additional energy dependence. However, it is possible that such a complicated energy dependence of the momentum spectra and excluded volumes of lightest hardons, i.e. pions and kaons, can be verified for highly accurate measurements, if the collective flow is correctly taken into account. The latter adjustment is tremendously complex because it is related to the freeze-out problem in relativistic hydrodynamics [24] or hydro-cascade approach [25]. Another possibility to study the effect of Lorentz contraction on the EOS properties is to incorporate them into transport models. The first steps in this direction have been made already in [26], but the approximation used in [26] is too crude.

It might be more realistic to incorporate the developed approach into effective models of nuclear/hadronic matter [6, 7, 8, 9] and check the obtained EOS on a huge amount of data collected by the nuclear physics of intermediate energies. Since the suggested relativization of the VdW EOS makes it softer at high densities, one can hope to improve the description of the nuclear/hadronic matter properties (compressibility constant, elliptic flow, effective nucleon masses e.t.c.) at low temperatures and high baryonic densities [27].

Also it is possible that the momentum spectra of this type can help to extend the hydrodynamic description into the region of large transversal momenta of hadrons ($p_T > 1.5 - 2$ GeV) which are usually thought to be too large to follow the hydrodynamic regime [28].

Another possibility to validate the suggested model is to study angular correlations of the hard core particles emitted from the neighboring regions and/or the enhancement of the particle yield of those hadrons occurring due to coalescence of the constituents with the short range repulsion. As shown above (also see Fig. 2), the present model predicts that the probability to find the neighboring particles with collinear velocities is higher than the one with non-collinear velocities. Due to this reason, the coalescence of particles with the parallel velocities should be enhanced. This effect amplifies if pressure is high and if particles are relativistic in the local rest frame. Therefore, it would be interesting to study the coalescence of any relativistic constituents with hard core repulsion (quarks or hadrons) at high pressures in a spirit of the recombination model of Ref. [29] and extend its results to lower transversal momenta of light hadrons. Perhaps, the inclusion of such an effect into consideration may essentially improve not only our understanding of the quark coalescence process, but also the formation of deuterons and other nuclear fragments in relativistic nuclear collisions. This subject is, however, outside the scope of the present work.

As a typical VdW EOS, the present model should be valid for the low particle densities. Moreover, our analysis of the limit $\mu/T \gg 1$ for fixed T leads to a surprisingly clear kinetic expression for the system's pressure (32). Therefore, it is possible that this low density result may provide a correct hint to study the relativistic analog of the dense packing problem.

Thus, it would be interesting to verify, whether the above approach remains valid for relativistic quantum treatment because there are several unsolved problems for the systems of relativistic bosons and/or fermions which, on one hand, are related to the problems discussed here and, on the other hand, may potentially be important for relativistic nuclear collisions and for nuclear astrophysics.

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Appendix A: Relativistic Excluded Volume

In order to study the high pressure limit, it is necessary to estimate the excluded volume of two ellipsoids, obtained by the Lorentz contraction of the spheres. In general, this is quite an involved problem. Fortunately, our analysis requires only the ultrarelativistic limit when the mean energy per particle is high compared to the mass of the particle. The problem can be simplified further since it is sufficient to find an analytical expression for the relativistic excluded volume with the collinear particle velocities because the configurations with the noncollinear velocities have larger excluded volume and, hence, are suppressed. Therefore, one can safely consider the excluded volume produced by two contracted cylinders (disks) having the same proper volumes as the ellipsoids. For this purpose the cylinder's height in the local rest frame is fixed to be $\frac{4}{3}$ of a sphere radius.

Let us introduce the different radii R_1 and R_2 for the cylinders, and consider for the moment a zero height for the second cylinder $h_2 = 0$ and non-zero height h_1 for the first one. Suppose that the center of the coordinate system coincides with the geometrical center of the first cylinder and the OZ-axis is perpendicular to the cylinder's base. Then the angle Θ_v between the particle velocities is also the angle between the bases of two cylinders. To simplify the expression for the pressure, the Lorentz frame is chosen to be the rest frame of the whole system.

In order to estimate the excluded volume we fix the particle velocities and transfer the second cylinder around the first cylinder while keeping the angle Θ_v fixed. The desired excluded volume is obtained as the volume occupied by the center of the second cylinder under these transformations. Considering the projection on the XOY plane (see Fig. 3.a), one should transfer the ellipse with the semiaxes $R_x = R_2 \cos(\Theta_v)$ and $R_y = R_2$ around the circle of radius R_1. We approximate it by the circle of the averaged radius $\langle R_{XOY} \rangle = R_1 + (R_x + R_y)/2 = R_1 + R_2(1 + \cos(\Theta_v))/2$. Then the first contribution to the excluded volume is the volume of the cylinder of the radius $\langle R_{XOY} \rangle$ and the height $h_1 = CC_1$ of the cylinder $OABC$ in Figs. 3.a and 3.b, i.e.,

$$v_I(h_1) = \pi \left(R_1 + \frac{R_2(1 + \cos(\Theta_v))}{2} \right)^2 h_1 \ . \tag{44}$$

Projecting the picture onto the XOZ plane as it is shown in Fig. 3.b, one finds that the translations of a zero width disk over the upper and lower bases of the first cylinder (the distance between the center of the disk and the base CA is, evidently, $CD_1 = R_2|\sin(\Theta_v)|$) generate the second conrtibution to the excluded volume

$$v_{II}(h_1) = \pi R_1^2\, 2\, R_2 |\sin(\Theta_v)|\ . \tag{45}$$

The third contribution follows from the translation of the disk from the cylinder's base to the cylinder's side as it is shown for the YOZ plane in Fig. 3.c. The area BB_1F is the part of the ellipse segment whose magnitude depends on the x coordinate. However, one can approximate it as the quarter of the disk area projected onto the YOZ plane and can get a simple answer $\pi R_2^2 |\sin(\Theta_v)|/4$. Since there are four of such transformations, and they apply for all x coordinates of the first cylinder (the length is $2\,R_1$), then the third contribution is

$$v_{III}(h_1) = \pi R_1^2\, 2\, R_1 |\sin(\Theta_v)|\ . \tag{46}$$

Collecting all the contributions, one obtains an estimate for the excluded volume of a cylinder and a disk

$$v_{2c}(h_1) = \pi \left(R_1 + R_2 \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 h_1 + 2\,\pi R_1 R_2 (R_1 + R_2) |\sin(\Theta_v)|\ . \tag{47}$$

The above equation, evidently, gives an exact result for a zero angle and arbitrary height of the first cylinder. Comparing it with the exact answer for $\Theta_v = \frac{\pi}{2}$

$$v_{2c}^E\left(h_1, \Theta_v = \frac{\pi}{2}\right) = R_1\,(\pi R_1 + 4\,R_2)\, h_1 + 2\,\pi R_1 R_2 (R_1 + R_2)\ , \tag{48}$$

one finds that the dominant terms (the second terms in (47) and (48)) again are exact, whereas the corresponding corrections, which are proportional to h_1 , are related to each other as ≈ 0.9897 (ratio of the approximate to exact values at $R_2 = R_1$). Therefore, Eq. (47) also gives a good approximation for the intermediate angles and small heights.

In order to get an expression for the non-zero height of the second cylinder we note that the expression for the excluded volume should be symmetric under the permutation of indexes 1 and 2. The lowest order correction in powers of the height comes from the contribution $v_I(h_1)$. Adding the symmertic contribution $v_I(h_2)$ to $v_{2c}(h_1)$ (47), one obtains the following result

$$\begin{aligned} v_{2c}(h_1, h_2) &= \pi \left(R_1 + R_2 \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 h_1 + \pi \left(R_2 + R_1 \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 h_2 \\ &+ 2\,\pi R_1 R_2 (R_1 + R_2) \sin(\Theta_v)\ . \end{aligned} \tag{49}$$

The above expression gives an exact result for a zero angle and arbitrary heights of cylinders. It also gives nearly exact answer for $\Theta_v = \frac{\pi}{2}$ in either limit h_1 or $h_2 \rightarrow 0$.

Choosing the heights to reproduce the proper volume for each of the Lorentz contracted spheres, one gets an approximation for the excluded volume of contracted spheres in ultra-relativistic limit

$$\begin{aligned} v_{12}^{Urel}(R_1, R_2) &= \frac{4}{3}\pi \frac{R_1}{\gamma_1} \left(R_1 + R_2 \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 + \frac{4}{3}\pi \frac{R_2}{\gamma_2} \left(R_2 + R_1 \cos^2\left(\frac{\Theta_v}{2}\right)\right)^2 \\ &+ 2\,\pi R_1 R_2 (R_1 + R_2) \sin(\Theta_v)\ . \end{aligned} \tag{50}$$

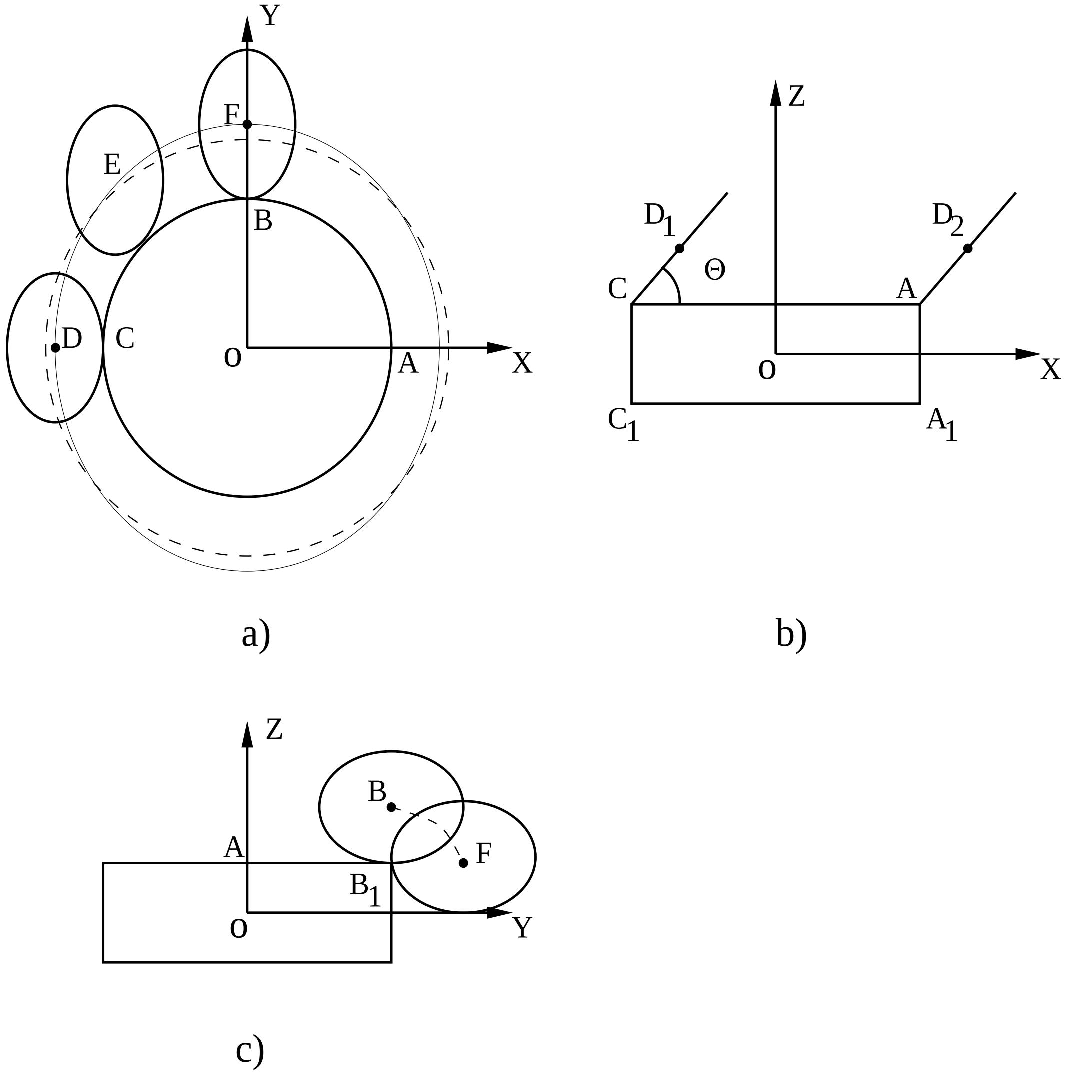

Fig. 3. Relativistic excluded volume derivation for relativistic cylinder $OABC$ and ultra-relativistic cylinder (disk) DC with radii R_1 and R_2, respectively. Θ is the angle between their velocities. Pictures a - c show the projections onto different planes. The transfer of the cylinder DC around the side of the cylinder $OABC$ is depicted in Fig. 3.a. The solid curve DEF corresponds to the exact result, whereas the dashed curve corresponds to the average radius approximation $\langle R_{XOY} \rangle = OA + (DC + BF)/2 = R_1 + R_2(1 + \cos(\Theta))/2$.
The transfer of the cylinder $DC = DC_1 = AD_2$ along the upper base of the cylinder $OABC = ACC_1A_1$ is shown in panel b. Its contribution to the excluded volume is a volume of the cylinder with the base $AC = 2R_1$ and the height $CD_1 \sin(\Theta) = R_2 \sin(\Theta)$. A similar contribution corresponds to the disk transfer along the lower base of the cylinder A_1C_1.
The third contribution to the relativistic excluded volume arises from the transformation of the cylinder $DC = BB_1 = FB_1$ from the upper base of the cylinder $OABC = AB_1O$ to its side, and it is schematically shown in Fig. 3.c. The area $BB_1F \approx \pi/4R_2^2 \sin(\Theta)$ is approximated as the one quarter of the area of the ellipse BB_1.

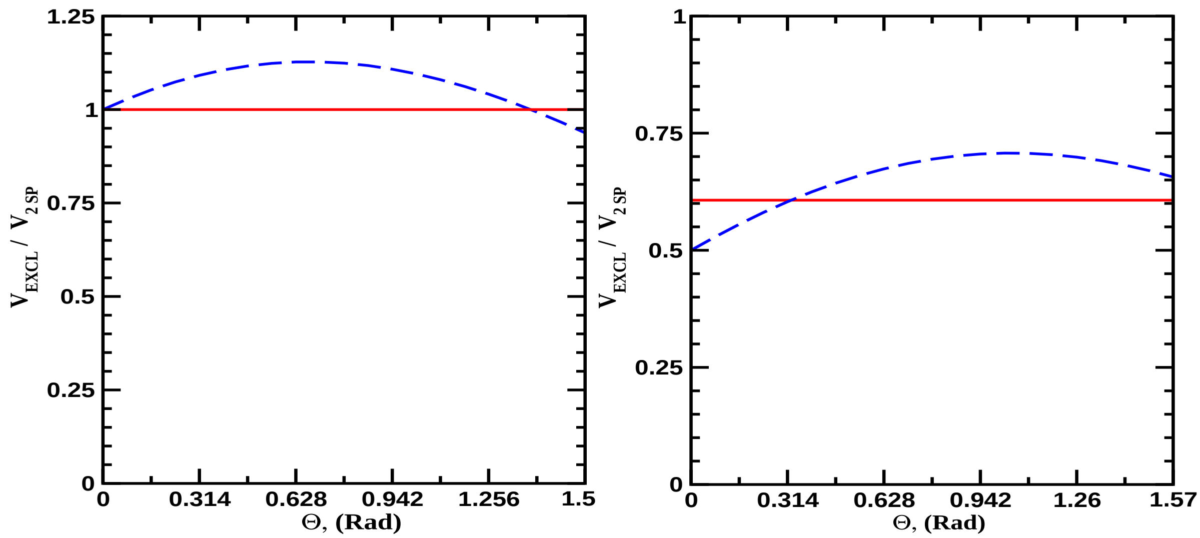

Fig. 4. Comparison of the relativistic excluded volume obtained by the approximative ultrarelativistic formula with the exact results. The left panel shows the quality of the approximation $V_{EXCL} \equiv v_{12}^{Urel}(R,R)$ (50) to describe the excluded volume of two nonrelativistic spheres V_{2SP} of the same radius R as a function of the spherical angle Θ. The right panel depicts the approximation to the excluded volume of the nonrelativistic sphere and disk. In both panels the solid curve corresponds to the exact result and the long dashed one corresponds to the ultrarelativistic approximation by two cylinders. The averaged ultrarelativistic excluded volume in the left panel is $\frac{\langle V_{EXCL}\rangle_\Theta}{V_{2SP}} \approx 1.065$. The corresponding averaged value for the right panel is $\frac{\langle V_{EXCL}\rangle_\Theta}{V_{2SP}} \approx 0.655$, which should be compared with the exact value $\frac{\langle V_{EXCL}\rangle_\Theta}{V_{2SP}} \approx 0.607$.

The corresponding γ_q-factors ($\gamma_q \equiv E(\mathbf{k}_q)/m_q$, $q = \{1,2\}$) are defined in the local rest frame of the whole system for particles of mass m_q. The last result is valid for $0 \leq \Theta_v \leq \frac{\pi}{2}$, to use it for $\frac{\pi}{2} \leq \Theta_v \leq \pi$ one has to replace $\Theta_v \longrightarrow \pi - \Theta_v$ in (50).

It is necessary to stress that the above formula gives a surprisingly good approximation even in nonrelativistic limit for the excluded volume of two spheres. For $R_2 = R_1 \equiv R$ one finds that the maximal excluded volume corresponds to the angle $\Theta_v = \frac{\pi}{4}$ and its value is $\max\{v_{12}^{Urel}(R,R)\} \approx \frac{36}{3}\pi R^3$, whereas the exact result for nonrelativistic spheres is $v_{2s} = \frac{32}{3}\pi R^3$, i.e., the ultrarelativistic formula (50) describes a nonrelativistic situation with the maximal deviation of about 10% (see the left panel in Fig. 4).

Eq. (50) also describes the excluded volume $v_{sd} = \frac{10+3\pi}{3} R^3$ for a nonrelativistic sphere and ultrarelativistic ellipsoid with the maximal deviation from the exact result of about 15% (see the right panel in Fig. 4).

In order to improve the accuracy of (50) for nonrelativistic case, we introduce a factor α to normalize the integral of the excluded volume (50) over the whole solid angle to the volume of two spheres

$$v^{Nrel}(R_1,R_2) = \alpha \; v_{12}^{Urel}(R_1,R_2)\,; \qquad \alpha = \frac{4\pi\,(R_1+R_2)^3}{3\int\limits_0^\pi d\Theta_v\,\sin(\Theta_v)\; v_{12}^{Urel}(R_1,R_2)\Big|_{\gamma_1=\gamma_2=1}}\,. \tag{51}$$

For the equal values of hard core radii and equal masses of particles the normalization factor reduces to the following value $\alpha \approx \frac{1}{1.0654}$, i.e., it compensates the most of the deviations discussed above. With such a correction the excluded volume (51) can be safely used for the nonrelativistic domain because in this case the VdW excluded volume effect is itself a correction to the ideal gas and, therefore, the remaining deviation from the exact result is of a higher order.

It is useful to have the relativistic excluded volume expressed in terms of 3-momenta

$$
\begin{aligned}
v_{12}^{Urel}(R_1,R_2) &= \frac{v_{01}}{\gamma_1}\left(1+R_2\frac{|\mathbf{k}_1||\mathbf{k}_2|+|\mathbf{k}_1\cdot\mathbf{k}_2|}{2\,R_1\,|\mathbf{k}_1||\mathbf{k}_2|}\right)^2+\frac{v_{02}}{\gamma_1}\left(1+R_1\frac{|\mathbf{k}_1||\mathbf{k}_2|+|\mathbf{k}_1\cdot\mathbf{k}_2|}{2\,R_2\,|\mathbf{k}_1||\mathbf{k}_2|}\right)^2 \\
&+ 2\,\pi R_1R_2(R_1+R_2)\frac{|\mathbf{k}_1\times\mathbf{k}_2|}{|\mathbf{k}_1||\mathbf{k}_2|}\ , \qquad (52)
\end{aligned}
$$

where v_{0q} denote the corresponding proper volumes $v_{0q}=\frac{4}{3}\pi R_q^3\,, \quad q=\{1,2\}$.

For the practical calculations it is necessary to express the relativistic excluded volume in terms of the three 4-vectors - the two 4-momenta of particles, $k_{q\,\mu}$, and the collective 4-velocity $u^\mu=\frac{1}{\sqrt{1-\mathbf{v}^2}}(1,\mathbf{v})$. For this purpose one should reexpress the gamma-factors and at least one of trigonometric functions in (50) in a covariant form

$$
\gamma_q=\frac{\sqrt{m^2+\mathbf{k}_q^2}}{m}=\frac{k_q^\mu\,u_\mu}{m}, \quad \cos\left(\Theta_v\right)=\frac{k_1^\mu\,u_\mu\;k_2^\nu\,u_\nu-k_1^\mu\,k_{2\,\mu}}{\sqrt{\left((k_1^\mu\,u_\mu)^2-m^2\right)\left((k_2^\mu\,u_\mu)^2-m^2\right)}}\ . \qquad (53)
$$

Using Eq. (53), one can express any trigonometric function of Θ_v in a covariant form.

References

[1] G. D. Yen and M.I. Gorenstein, Phys. Rev. **C 59** (1999) 2788.

[2] P. Braun-Munzinger, I. Heppe and J. Stachel, Phys. Lett. **B 465** (1999) 15.

[3] K. A. Bugaev, M. I. Gorenstein, H. Stöcker and W. Greiner. Phys. Lett. **485** (2000) 121.

[4] G. Zeeb, K. A. Bugaev, P. T. Reuter and H. Stöcker, nucl-th/0209011.

[5] R. Venugopalan and M. Prakash, Nucl. Phys. **A546** (1992) 718.

[6] D. H. Rischke, M. I. Gorenstein, H. Stöcker and W. Greiner, Z. Phys. **C51** (1991) 485.

[7] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345; Phys. Rev. **124** (1961) 246.

[8] P. A. M. Guichon, Phys. Lett. **B 200** (1988) 235.

[9] P. Papazoglou *et. al.*, Phys. Rev. **C 57** (1998) 2576.

[10] S. Datta, F. Karsch, P. Petreczky and I. Wetzorke, hep-lat/0208012.

[11] M. Asakawa and T. Hatsuda, Nucl. Phys. **A 715** (2003) 863c.

[12] see also discussions and references in E. V. Shuryak, Prog. Part. Nucl. Phys. **53** (2004) 273; E. V. Shuryak and I. Zahed, Phys. Rev. **C 70** (2004) 021901; E. V. Shuryak, Nucl. Phys. A **774** (2006) 387; hep-ph/0510123.

[13] E. V. Shuryak and I. Zahed, Phys. Rev. **D 70** (2004) 054507; hep-ph/0403127.

[14] M. Mannarelli and R. Rapp, hep-ph/0505080.

[15] K. A. Bugaev, Phys. Rev. **C 76** (2007) 014903.

[16] Q. R. Zhang, Z. Phys. **A 353** (1995) 345.

[17] J. E. Mayer and M. Goeppert-Mayer,"Statistical Mechanics" (1977)

[18] K. A. Bugaev, J. Phys. **G 28** (2002) 1981.

[19] M. I. Gorenstein, K. A. Bugaev and M. Gazdzicki, Phys. Rev. Lett. **88** (2002) 132301; K. A. Bugaev, M. Gazdzicki, M. I. Gorenstein, Phys. Lett. **B 544** (2002) 127.

[20] see, for instance, A. Münster: "Statistical Thermodynamics" Vol. II, Springer-Verlag, Heidelberg 1974.

[21] for instance, see M. Gyulassy, Prog. Part. Nucl. Phys. **15** (1985) 403 and references therein.

[22] J. D. van der Waals Z. Phys. Chem. **5** (1889) 133.

[23] M. I. Gorenstein, A. P. Kostyuk and Y. D. Krivenko J. Phys. **G 25** (1999) L75.

[24] K. A. Bugaev, Nucl. Phys. **A 606** (1996) 559; K. A. Bugaev and M. I. Gorenstein, nucl-th/9903072 (1999) 70 p.

[25] K. A. Bugaev, Phys. Rev. Lett. **90** (2003) 252301; Phys. Rev. **C70** (2004) 034903 and references therein.

[26] A. B. Larionov, O. Buss, K. Gallmeister and U. Mosel, arXiv:nucle-th/0704.1785.

[27] P. Danielewicz, nucl-th/0512009 and references therein.

[28] U. W. Heinz, nucl-th/0504011 and references therein.

[29] R. J. Fries, B. Muller, C. Nonaka and S. A. Bass, Phys. Rev. C **68** (2003) 044902 [nucl-th/0306027].